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Number operators in a general quon algebra

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Abstract. The existence of the number operators N_k in a general quon algebra is proved and a general construction of N_k is presented. A new general simple structure of N_k is proved when $|q_{ij}| < 1$, $\forall i, j \in S$. General recurrent relations for the corresponding coefficients are constructed and solved in special cases. These solutions are discussed and a conjecture for a general solution is proposed. Cases in which some or all $|q_{ij}| = 1$ are discussed.

1. Introduction

Quonic intermediate statistics [1, 2], interpolating between Bose and Fermi statistics, are examples of infinite statistics in which any representation of the symmetric group can occur. This offers a possibility for a small violation of the Pauli exclusion principle, at least in non-relativistic theory [2, 3]. A more general quon algebra with infinite statistics, interpolating between Bose, Fermi, para-Bose, para-Fermi and anyonic statistics was proposed and analysed in [4]. The deformation parameter is not global but is an arbitrary function of distance (momentum). In $2 + 1$ dimensions, anyons represent the limiting case of quon particles [5]. In $3 + 1$ dimensions, no generalized statistics is possible from dynamics in curved spacetime for a global deformation parameter [6]. However, generalized statistics is possible, at least in principle, for a local deformation parameter [7]. It seems interesting to further analyse a general quon algebra and its consequences, particularly in lower dimensions, for example, in connection with anyons in $2 + 1$ dimensions.

In this paper we show that number operators exist in the general quon algebra [4], and present a general construction of number operators when $|q_{ij}| < 1$. We prove the general simple structure of the number operator [4] and construct recurrent relations for the corresponding coefficients in all orders. Solutions of these equations are discussed (up to third order) and a general solution is conjectured.

2. Existence of number operators

Let us start with the general associative quon algebra [4]

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij} \qquad \forall i, j \in S \qquad q_{ij}^* = q_{ji} \tag{1}$$

where $a_i^\dagger$ is the Hermitian conjugate of a_i (and vice versa). The indices i, j belong to some lattice S. No commutation relation between a_i and a_j exists if $|q_{ij}| \neq 1$ for $\forall i, j \in S$ [1, 2, 4]. The Fock-like space of all states is positive definite [4, 8]. In order to construct a

Fock-like representation of the general quon algebra, we assume that the unique vacuum $|0\rangle$ (and its dual $\langle 0|$) exists. The vacuum conditions are

$$\langle 0|a_i^\dagger = 0 \qquad a_i|0\rangle = 0 \qquad \forall i \in S \qquad \langle 0|0\rangle = 1. \tag{2}$$

By definition, the operators $a_i^\dagger$ acting on the vacuum create one-particle states $a_i^\dagger|0\rangle$. Normalizing these one-particle states and using equation (1), one obtains (for $n, m \in N$)

$$\langle 0|(a_i)^n(a_j^\dagger)^m|0\rangle = [n]_{q_{ii}}!\delta_{nm}\delta_{ij} \tag{3}$$

where

$$\begin{aligned}[n]! &= [n]\cdot[n-1]\cdots[1]\\ [n]_{q_{ii}} &= \frac{(q_{ii})^n - 1}{q_{ii}-1}.\end{aligned} \tag{4}$$

We point out that the general matrix element

$$\mathcal{A}_{i_1\cdots i_n;j_1\cdots j_m} = \langle 0|a_{i_n}\cdots a_{i_1}a_{j_1}^\dagger\cdots a_{j_m}^\dagger|0\rangle \tag{5}$$

vanishes unless $n = m$ and the indices $(i_1\cdots i_n)$ and $(j_1\cdots j_m)$ are equal up to permutation. The matrix element $\mathcal{A}_{\pi;\sigma}$, where π and σ are two permutations of $i_1\cdots i_n$, where i_j are mutually different, is given in [4].

Owing to equations (1) and (2) and to the orthogonality conditions for these states, the n-particle state is any state of the form $a_{i_1}^\dagger a_{i_2}^\dagger\cdots a_{i_n}^\dagger|0\rangle$. Hence, the *total* number operator in the Fock space exists. In the same way, equations (1) and (2) and the orthogonality conditions for these states allow us to define the number operator N_k counting the $a_k^\dagger$ operators in any n-particle state of the form $a_{i_1}^\dagger a_{i_2}^\dagger\cdots a_{i_n}^\dagger|0\rangle$, $i_1, i_2, \cdots i_n \in S$, in the Fock space. Namely,

$$N_k\,(a_{i_1}^\dagger a_{i_2}^\dagger\cdots a_{i_n}^\dagger|0\rangle) = \sum_{\alpha=1}^{n}\delta_{ki_\alpha}(a_{i_1}^\dagger\cdots a_{i_\alpha}^\dagger\cdots a_{i_n}^\dagger|0\rangle). \tag{6}$$

Hence, the number operator N_k, $k \in S$, exists and is a Hermitian operator by definition. We note that the norm of any linear combination of n-particle states is positive definite when $|q_{ij}| < 1$, $\forall i, j \in S$ [4,8].

3. Construction of the number operator

From equation (6) it follows that

$$[N_k, a_l] = -\delta_{kl}a_k \qquad \forall k, l \in S \qquad [N_k, a_l^\dagger] = \delta_{kl}a_k^\dagger. \tag{7}$$

When $q_{ij} = +1$ (or -1), $\forall i, j \in S$, i.e. in the Bose (or Fermi) case, the number operator N_k is given by $N_k = a_k^\dagger a_k$. Hence we start our construction of the operator N_k when $|q_{ij}| < 1$, $\forall i, j \in S$, with the term $a_k^\dagger a_k$. The general form of N_k is given by a series expansion in the creation and annihilation operators,

$$N_k = c_k a_k^\dagger a_k + \sum_{n=1}^{\infty}\sum_{(i_1\cdots i_n)}(X_{ki_1\cdots i_n})^\dagger Y_{ki_1\cdots i_n} \tag{8}$$

where

$$\begin{aligned}X_{ki_1\cdots i_n} &= \sum_{\pi\in S_{n+1}/St} x_{\pi(ki_1\cdots i_n)}\pi\cdot(a_k a_{i_1}\cdots a_{i_n})\\ Y_{ki_1\cdots i_n} &= \sum_{\sigma\in S_{n+1}/St} y_{\sigma(ki_1\cdots i_n)}\sigma\cdot(a_k a_{i_1}\cdots a_{i_n})\end{aligned} \tag{9}$$

where we define

$$\sigma \cdot (a_{i_1} a_{i_2} \cdots a_{i_{n+1}}) \doteq a_{i_{\sigma(1)}} a_{i_{\sigma(2)}} \cdots a_{i_{\sigma(n+1)}}$$
$$\sigma \cdot (i_1 i_2 \cdots i_{n+1}) \doteq i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(n+1)}.$$

The summation in equation (9) has been performed over all different permutations of indices $k i_1 \cdots i_n$. Note that, generally, there are $\frac{(n+1)!}{s_1! \cdots s_a!}$, $(\sum_{\alpha=1}^{a} s_\alpha = n+1)$ such permutations, where $s_1 \cdots s_a$ are multiplicities of the indices appearing in the sequence $k i_1 \cdots i_n$. S_{n+1} denotes the symmetric group of $(n+1)$ elements and x_π and y_σ are complex coefficients.

Note that the particular terms of the form $X^\dagger \cdot Y$ in the expansion for N_k are not necessarily Hermitian operators, but for any fixed $n \in N$ their sum corresponding to n is a Hermitian operator.

Inserting N_k, equation (8), into $[N_k, a_l] = -\delta_{kl} a_k$ and using equation (1), we obtain

$$c_k \, (a_k^\dagger a_k a_l - q_{lk} a_k^\dagger a_l a_k) + \sum_{n=1}^{\infty} \sum_{(i_1 \cdots i_n)} [(X_{k i_1 \cdots i_n})^\dagger Y_{k i_1 \cdots i_n} \, a_l$$
$$-a_l \, (X_{k i_1 \cdots i_n})^\dagger \, Y_{k i_1 \cdots i_n}] = (c_k - 1) \, a_k \, \delta_{kl}. \tag{10}$$

All terms in equation (10) are of the type $a_{i_1}^\dagger \cdots a_{i_n}^\dagger a_{j_1} \cdots a_{j_{n+1}}$, $n \in N$. The main idea is to collect the terms of the same type and equate them to zero. Using the fact that for $|q_{ij}| < 1, \forall i, j \in S$, the set of all monomials $(a_{i_n}^\dagger \cdots a_{i_1}^\dagger a_{j_1} \cdots a_{j_{n+1}})$, $i_\alpha, j_\beta \in S$, is linearly independent [1, 2, 4], all the corresponding coefficients in equation (10) should vanish. In this way we end up with a set of linear equations for the corresponding coefficients.

To proceed in this way, we first write

$$a_l \, (X_{k i_1 \cdots i_n})^\dagger = \widehat{a_l (X}_{k i_1 \cdots i_n})^\dagger + q_{lk} \, q_{l i_1} \cdots q_{l i_n} (X_{k i_1 \cdots i_n})^\dagger a_l \tag{11}$$

where the first term on the RHS denotes the sum of all possible contractions and is of the type $(a^\dagger)^n$, whereas the second term on the RHS is of the type $(a^\dagger)^{n+1} a$. For example,

$$\widehat{a_l (a}_{i_1}^\dagger \cdots a_{i_n}^\dagger) = \sum_{\alpha=1}^{n} q_{l i_1} \cdots q_{l i_{\alpha-1}} a_{i_1}^\dagger \cdots \not{a}_{i_\alpha}^\dagger \cdots a_{i_n}^\dagger \delta_{l i_\alpha} \tag{12}$$

where a slash denotes the omission of the corresponding creation operator. In the lowest order (terms of type a) we find from (10) that

$$(c_k - 1) a_k = 0 \quad \Rightarrow c_k = 1 \qquad \forall k \in S \tag{13}$$

and in the next order,

$$a_k^\dagger (a_k a_l - q_{lk} a_l a_k) = \sum_{i_1} \widehat{a_l (X}_{k i_1})^\dagger Y_{k i_1}. \tag{14}$$

Equation (14) can be decomposed into two relations:

$$\begin{aligned} Y_{k i_1} &= a_k a_{i_1} - q_{i_1 k} \, a_{i_1} a_k \\ \widehat{a_l (X}_{k i_1})^\dagger &= a_k^\dagger \delta_{l i_1}. \end{aligned} \tag{15}$$

The solution of the second of equations (15) is

$$(X_{k i_1})^\dagger = \frac{1}{1 - |q_{k i_1}|^2} (Y_{k i_1})^\dagger \qquad |q_{k i_1}| \neq 1. \tag{16}$$

If $|q_{kl}| = 1$, the associative quon algebra, equation (1), implies the following relation between a_k and a_l [4, 7]:

$$a_k a_l - q_{lk} \, a_l a_k = 0. \tag{17}$$

Equation (14) reduces to

$$\sum_{i_1} \widehat{a_l(X_{ki_1})^\dagger} Y_{ki_1} = 0 \qquad Y_{kl} = 0. \tag{18}$$

Hence $(X_{kl})^\dagger Y_{kl} = 0$. If $|q_{kk}| = 1$, then $q_{kk} = \pm 1$, and the same argument applies. The successive recurrent relations for the general nth order, which follow from equations (10) and (11), are

$$\sum_{(i_1,\cdots i_{n+1})} (X_{ki_1\cdots i_n})^\dagger [Y_{ki_1\cdots i_n} a_{i_{n+1}} - q_{i_{n+1}k}\, q_{i_{n+1}i_1} \cdots q_{i_{n+1}i_n} a_{i_{n+1}} Y_{ki_1\cdots i_n}]\, \delta_{li_{n+1}}$$

$$= \sum_{(i_1,\cdots i_{n+1})} \widehat{a_l(X_{ki_1\cdots i_{n+1}})^\dagger}\, Y_{ki_1\cdots i_{n+1}}. \tag{19}$$

From equation (19) we obtain two relations when $|q_{ij}| < 1$, for $\forall i, j \in S$:

$$Y_{ki_1\cdots i_{n+1}} = Y_{ki_1\cdots i_n} a_{i_{n+1}} - q_{i_{n+1}k} q_{i_{n+1}i_1} \cdots q_{i_{n+1}i_n} a_{i_{n+1}}\, Y_{ki_1\cdots i_n} \tag{20}$$

and

$$\widehat{a_l(X_{ki_1\cdots i_{n+1}})^\dagger} = (X_{ki_1\cdots i_n})^\dagger \delta_{li_{n+1}}. \tag{21}$$

In order to solve equation (21), we use an important relation which can be easily proved by induction,

$$\widehat{a_l(Y_{ki_1\cdots i_n})^\dagger} = \sum_{j=1}^{n} d_j^{l(k;n)} (Y_{ki_1\cdots \not{i_j} \cdots i_n})^\dagger\, \delta_{li_j} \tag{22}$$

where the slashed $\not{i}_j$ denotes the omission of the index i_j, and the coefficients $d_j^{l\,(k;n)}$ are given by

$$d_j^{l\,(k,i_1\cdots i_n)} = q_{li_n} \cdots q_{li_{j+1}} (1 - |q_{li_{j-1}} \cdots q_{li_1}\, q_{lk}|^2). \tag{23}$$

Using this result and comparing it with equation (21), we can generally write

$$(X_{ki_1\cdots i_n})^\dagger = \sum_{\pi \in S_n/St} c_{k\pi(i_1\cdots i_n),ki_1\cdots i_n} [Y_{k\pi(i_1\cdots i_n)}]^\dagger \tag{24}$$

where the summation is over all different permutations of $i_1, \cdots i_n$.

Hence, the general simple structure of the number operator N_k (when $|q_{ij}| < 1$, $\forall i, j \in S$) is

$$N_k = a_k^\dagger a_k + \sum_{n=1}^{\infty} \sum_{(i_1\cdots i_n)} \sum_{\pi \in S_n/St} c_{k\pi(i_1\cdots i_n),ki_1\cdots i_n} \cdot [Y_{k\pi(i_1\cdots i_n)}]^\dagger Y_{ki_1\cdots i_n}. \tag{25}$$

Inserting equation (24) into equation (21) and using equations (22),(23), we obtain

$$\frac{n!}{s_1! \cdots s_a!} \qquad \left(\sum_{\alpha=1}^{a} s_\alpha = n\right)$$

linear equations for the same number of unknown coefficients $c_{k\pi(i_1\cdots i_n),ki_1\cdots i_n}$. They are

$$\sum_{j=1}^{n} d_j^{l(k,\Lambda(j,l,\pi)(1\cdots n))} c_{k\Lambda(j,l,\pi)(1\cdots n);k1\cdots n} = c_{k\pi(1\cdots n-1);k1\cdots(n-1)}\, \delta_{ln}$$

$$l = 1, 2, \cdots n \tag{26}$$

where

$$\Lambda(j,l,\pi) = \varepsilon_j^l \cdot \pi \cdot \eta_l$$
$$\eta_l(i_1, \cdots i_n) = (i_1, \cdots i_{l-1}, i_{l+1} \cdots i_n) = (i'_1, \cdots i'_{n-1})$$
$$\varepsilon_j^l \cdot \pi(i'_1, \cdots i'_{n-1}) = (i'_{\pi(1)} \cdots i'_{\pi(j-1)} i_l i'_{\pi(j+1)} \cdots i'_{\pi(n-1)}).$$

($1, 2, \cdots n$ in equation (26) denote abbreviations for $i_1 \cdots i_n \in S$). The coefficients $d_j^{l(k;n)}$ are defined by equation (23).

The following hermiticity relation holds:

$$c^*_{k\pi(i_1\cdots i_n),ki_1\cdots i_n} = c_{ki_1\cdots i_n,k\pi(i_1\cdots i_n)}. \tag{27}$$

4. Special cases

Let us write equations (26) and the corresponding solutions for $n = 1, 2, 3$ and for $|q_{ij}| < 1$, $\forall i, j \in S$.

(i) Case $n = 1$

$$\begin{aligned} d_1\ c_{ki_1,ki_1} &= c_k = 1 \\ (1 - |q_{ki_1}|^2)c_{ki_1,ki_1} &= 1 \\ c_{ki_1,ki_1} &= \frac{1}{1 - |q_{ki_1}|^2}. \end{aligned} \tag{28}$$

(ii) Case $n = 2$

When $i_1 \neq i_2$, equations (26) are

$$\begin{aligned} d_2^{2(k,12)}\ c_{k12,k12} + d_1^{2(k,21)}\ c_{k21,k12} &= c_{k1,k1} \\ d_1^{1(k,12)}\ c_{k12,k12} + d_2^{1(k,21)}\ c_{k21,k12} &= 0 \end{aligned} \tag{29}$$

where $d_j^{l(k,i_1i_2)}$ are given by equation (23); they are

$$\begin{aligned} d_1^{1(k,12)} &= q_{12}(1 - |q_{1k}|^2) \\ d_1^{2(k,21)} &= q_{21}(1 - |q_{2k}|^2) \\ d_2^{1(k,21)} &= (1 - |q_{1k}q_{12}|^2) \\ d_2^{2(k,12)} &= (1 - |q_{2k}q_{21}|^2). \end{aligned} \tag{30}$$

The solution of the equation (29) is

$$\begin{aligned} c_{ki_1i_2,ki_1i_2} &= \frac{1 - |q_{i_1i_2}\ q_{ki_1}|^2}{\Delta_3} = \mathcal{A}^{-1}_{ki_1i_2,ki_1i_2} = \mathcal{A}^{-1}_{k,id;k,id}(k, i_1, i_2) \\ c_{ki_2i_1,ki_1i_2} &= -\frac{q_{i_1i_2}(1 - |q_{ki_1}|^2)}{\Delta_3} = \mathcal{A}^{-1}_{ki_1i_2,ki_2i_1} = \mathcal{A}^{-1}_{k,id;k,\pi(1\leftrightarrow 2)}(k, i_1, i_2) \end{aligned} \tag{31}$$

where

$$\Delta_3 = (1 - |q_{i_1i_2}|^2)(1 - |q_{ki_1}|^2)(1 - |q_{i_1i_2}\ q_{ki_1}q_{ki_2}|^2) \neq 0. \tag{32}$$

If $i_1 = i_2$, equations (26) reduces to only one equation for $c_{kii,kii}$:

$$(d_2^{i(k,ii)} + d_1^{i(k,ii)})c_{kii,kii} = c_{ki,ki} \tag{33}$$

where

$$\begin{aligned} d_2^{i(k,ii)} &= 1 - |q_{ik}\ q_{ii}|^2 \\ d_1^{i(k,ii)} &= q_{ii}\ (1 - |q_{ik}|^2). \end{aligned} \tag{34}$$

Then

$$c_{kii,kii} = \frac{1}{(1 + q_{ii})(1 - q_{ii}\ |q_{ik}|^2)(1 - |q_{ik}|^2)} = \mathcal{A}^{-1}_{kii,kii}. \tag{35}$$

(iii) Case $n = 3$

Here we give solutions of equations (26) when $q_{ij} = q$, $\forall i, j \in S$, $q \in \boldsymbol{R}$.

If $i_1 \neq i_2 \neq i_3 \neq i_1$, they are

$$
\begin{aligned}
c_{k123,k123} &= \frac{(1+q^2)^2(1+q^4) - q^4}{\Delta_4} = \mathcal{A}^{-1}_{k123,k123} \\
c_{k231,k123} &= c_{k312,k123} = -\frac{q^4}{\Delta_4} = \mathcal{A}^{-1}_{k123,k231} \\
c_{k132,k123} &= -\frac{q\,(1+q^2)(1+q^4)}{\Delta_4} = \mathcal{A}^{-1}_{k123,k213} \\
c_{k213,k123} &= -\frac{q(1+q^6)}{\Delta_4} = \mathcal{A}^{-1}_{k123,k132} \\
c_{k321,k123} &= \frac{q^3(1+q^2)}{\Delta_4} = \mathcal{A}^{-1}_{k123,k321}.
\end{aligned}
\tag{36}
$$

where

$$
\Delta_4 = (1-q^2)(1-q^6)(1-q^{12}). \tag{37}
$$

If two of the indices $i_1 i_2 i_3$ are equal and the third is different, equations (26) reduce to three equations for $c_{kiij,kiij}$, $c_{kiji,kiij}$ and $c_{kjii,kiij}$. If the indices $i_1 i_2 i_3$ are equal, there is only one equation for $c_{kiii,kiii}$.

For general $n \in \boldsymbol{N}$ and $q_{ij} = q$, $\forall i, j \in S$, $q \in \boldsymbol{R}$, the coefficients $d_j^{l(k;n)}$ are

$$
d_j^{l(k;n)} = q^{n-j}(1-q^{2j}).
$$

The following symmetry relation can be proved:

$$
c_{k\pi\cdot(i_1\cdots i_n),ki_1\cdots i_n} = c_{k\pi^{-1}\cdot(i_1\cdots i_n),ki_1\cdots i_n} \tag{38}
$$

and the following sum rule holds:

$$
\sum_{\pi \in S_n/St} c_{k\pi\cdot(i_1\cdots i_n),ki_1\cdots i_n} = \frac{(1-q)^{n+1}}{1-q^{n+1}} \prod_{\alpha=1}^{n} \frac{1}{(1-q^\alpha)^2} = c_{k\underbrace{i\cdots i}_{n},k\underbrace{i\cdots i}_{n}}. \tag{39}
$$

5. Concluding remarks

We conclude with a few remarks.

(a) When $|q_{ij}| < 1$, $\forall i, j \in S$, our general construction leads to the recurrent relations (26). They have a unique non-trivial solution owing to the fact that the number operator N_k exists and that the states $a^\dagger_{\pi(i_1)} \cdots a^\dagger_{\pi(i_n)}|0\rangle$ are linearly independent for different permutations $\pi \in S_n$ [4].

If $|q_{ij}| = 1$ for some $i, j \in S$, equations (26) have no solution, i.e. the determinant of the system vanishes. For example, consider equation (31) when $|q_{i_1 i_2}| = 1$ or $|q_{ik}| = 1$. However, the number operator N_k still exists. The explanation is simple. We remind ourselves that the commutation relation between a_i and a_j appears when $|q_{ij}| = 1$, namely, $||(a_i a_j - q_{ji} a_j a_i)^\dagger|0\rangle||^2 = 0$, [4, 7]. Hence there is a reduction in the number of linearly independent states and our recurrent relations (26) are not valid in this case (solutions diverge).Nevertheless, in the limit when $|q_{ij}| \to 1$ and the determinant of the system (26) tends to zero, any matrix element of the operator

$$
\sum_{\pi \in S_n/St} c_{k\pi(i_1\cdots i_n),ki_1\cdots i_n} [Y_{k\pi\cdot(i_1\cdots i_n)}]^\dagger Y_{k(i_1\cdots i_n)} \tag{40}
$$

is finite. Moreover, in the limit when $|q_{ij}| \to 1$ for every pair of given indices $k, i_1, \cdots, i_n$, every matrix element of operator (40) tends to zero, and in this case we can omit all such operators (40) in expansion (25).

When $|q_{ij}| = 1$, $\forall i, j \in S$, (anyonic-like oscillators), the commutation relations are $a_i a_j - q_{ij}^* a_j a_i = 0$, $\forall i, j \in S$ and the number operator is simply $N_k = a_k^\dagger a_k$ [4,7].

(b) Analysing the solutions of equations (26), we have found [4] that the coefficients $c_{k\pi\cdot(i_1\cdots i_n),ki_1\cdots i_n}$ can be written as

$$c_{k\pi\cdot(i_1\cdots i_n),ki_1\cdots i_n} = \mathcal{A}^{-1}_{k,i_1\cdots i_n;k\sigma(i_1\cdots i_n)} \tag{41}$$

where $\mathcal{A}$ is defined by equation (5) and $\pi, \sigma \in S_n/St$. Note that $\mathcal{A}$ and $\mathcal{A}^{-1}$ are Hermitian matrices. Using the general structure of equations (1), (2), (5), one can show that the following relations hold:

$$\mathcal{A}^{-1}_{k,id;k,\pi}(k, i_1, \cdots, i_n) = \sum_{\sigma\in S_n} \mathcal{A}^{-1}_{k,id;k,\sigma}(k, i_1, \cdots, i_n) \tag{42}$$

where $\pi \in S_n/St$ and sum runs over $s_1!\cdots s_a!$, $(\sum_{\alpha=1}^{a} s_\alpha = n)$ different permutations σ, $\sigma \in S_n$, which satisfy $\sigma(i_1 \cdots i_n) = \pi(i_1 \cdots i_n)$.

We conjecture that formula (41) is true in all orders and represents a general solution of equation (26) [4], when $|q_{ij}| < 1$, $\forall i, j \in S$.

(c) We shall summarize the main results of this paper. We have proved the existence of the number operator N_k in the general quon algebra. We have presented a general construction of N_k when $|q_{ij}| < 1$, $\forall i, j \in S$. We have proved a new general simple structure of N_k, equation (25), and constructed general recurrent relations for the corresponding coefficients, equations (26). When $q_{ij} = q$, $-1 < q < 1$, our general construction differs from that proposed by Zagier [8], Stanciu [9] and Møller [10].

Our results are in agreement with those of [8–10]. For example, the result of Zagier can be written as

$$N_k = a_k^\dagger a_k + \frac{1}{1-q^2}\sum_i [Y_{k,i}]^\dagger Y_{k,i} + \frac{1}{(1-q^2)(1-q^6)}\sum_{i_1,i_2}[(1+q^2)Y_{k,i_1i_2} - qY_{k,i_2i_1}]^\dagger Y_{k,i_1i_2}. \tag{43}$$

(d) Finally, let us consider non-relativistic field theory of quons [2]. To construct the operators for the energy, momentum, angular momentum etc in terms of the annihilation and creation operators, one needs to construct a set of number operators, N_i, which obey the usual commutation relations (7). Then the energy operator, for example, is

$$E = \sum_i \epsilon_i N_i \tag{44}$$

where ϵ_i is the single particle energy.

Transition operators N_{ij}

$$[N_{ij}, a_k] = -a_i \delta_{jk} \tag{45}$$

can be constructed in the same way as we did for the number operators. Transition operators for infinite statistics are discussed in [2].

Usual number operators N_i (7) exist if there are no relations between a_i^m, a_j^n, $m, n \in N$. For general quon algebra, $|q_{ij}| < 1$, $\forall i, j \in S$, there are no such relations and hence number operators N_i exist.

Let us denote a quon field in position space by $\psi(x)$. To write a Hamiltonian for two-body interactions of identical particles which violate (para)Fermi, (para)Bose and anyonic statistics (by a possible small amount), it is convenient to work in position space. Then the analogue of the transition operator is the off-diagonal one-body operator $\rho_1(x; x')$ which obeys the relations

$$[\rho_1(x; x'), \psi^\dagger(y)] = \delta(x' - y)\psi^\dagger(x)$$
$$\rho_1(x; y)|0\rangle = 0. \tag{46}$$

The operator ρ_1 suffices for the one-body operators, such as the kinetic energy; however, for two-body operators relevant to potential interactions one needs an analogous operator $\rho_2(x, y; x', y')$ which obeys [2]:

$$[\rho_2(x, y; x', y'), \psi^\dagger(z)] = \delta(x' - z)\psi^\dagger(x)\rho_1(y; y') + \delta(y' - z)\psi^\dagger(y)\rho_1(x; x'). \tag{47}$$

Our method for constructing number operators N_i can be extended to $\rho_1(x; x')$ and $\rho_2(x, y; x', y')$ operators as well.

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